MIDTERM II

Université d'Ottawa • University of Ottawa

MAT 1320 D

WEDNESDAY, MARCH 18, 2009 **Name:** Solutions.

Student Number:

Read all of the following information before starting the exam:

- Only basic scientific calculators (non-programmable, non-graphing, no differentiation or integration capability) are allowed on the exam.
- Notebooks, notes, cheating sheets, and books are NOT permitted.
- Students must present their student cards if asked.
- Students may not leave until one hour after the examination has begun.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- **Circle or otherwise indicate your final answers.**
- This test has **SEVEN** problems and is worth **100** points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Question	1	2	3	4	5	6	7	TOTAL
Points	10	15	15	15	15	15	15	100
Mark								

1. (*10 points*) Find the derivative of $y = x^{\cos x}$.

(sol) $\ln y = \ln x^{\cos x}$

$\Rightarrow \ln y = (\cos x) \ln x$

$\Rightarrow \dfrac{y'}{y} = (-\sin x) \ln x + \cos x \cdot \dfrac{1}{x}$

$\Rightarrow y' = y\left[-\sin x \cdot \ln x + \dfrac{\cos x}{x}\right]$

$\therefore y' = x^{\cos x}\left(\dfrac{\cos x}{x} - \sin x \cdot \ln x\right)$

[redacted]

2. (*15 points*) Find an equation of the tangent to the curve $x^2 + xy + 2y^3 = 4$ at the point $(-2, 1)$.

(sol). To find the slope of the tangent we differentiate the given EQN implicitly w.r.t x.

$$2x + y + xy' + 6y^2y' = 0$$

$$\Rightarrow (x + 6y^2)\, y' = -(2x + y)$$

$$\Rightarrow y' = -\frac{(2x+y)}{(x+6y^2)}$$

Thus, at the pt $(-2, 1)$, the slope of the tangent line is

$$y'\Big]_{(x,y)=(-2,1)} = -\frac{2(-2)+1}{-2+6\cdot 1^2} = -\frac{-4+1}{-2+6} = \underline{\frac{3}{4}}$$

and its equation is

$$\underline{y - 1 = \frac{3}{4}(x+2)} \qquad \text{or} \qquad y = \frac{3}{4}x + \frac{10}{4}$$

$$(3x - 4y = -10)$$

3. (*15 points*) Find the linear approximation of the function $f(x) = \sqrt{1-x}$ at $a = 0$ and use it to approximate the number $\sqrt{0.97}$.

(sol) $f(x) = \sqrt{1-x} \Rightarrow f'(x) = \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) = \dfrac{-1}{2\sqrt{1-x}}$

so, $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

$$\sqrt{1-x} \approx f(0) + f'(0)(x-0) = 1 - \frac{x}{2}.$$

$$\therefore \sqrt{1-x} \approx 1 - \frac{x}{2}.$$

Thus, $\sqrt{0.97} = \sqrt{1-0.03} \approx 1 - \dfrac{0.03}{2} = 0.985$

Therefore, $\underline{\sqrt{0.97} \approx 0.985.}$

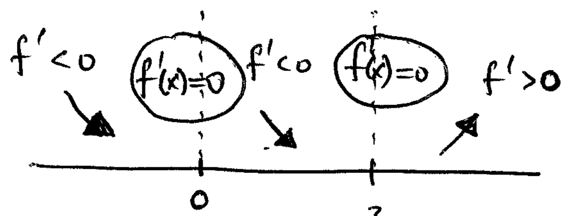
[redacted]

4. (*15 points*) Let $f(x) = x^4 - 4x^3 + 10$.

a. (*5 pts*) Find the intervals on which f is increasing or decreasing.

(sol) $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f'(x) = 0$ when $x = 0, 3$.

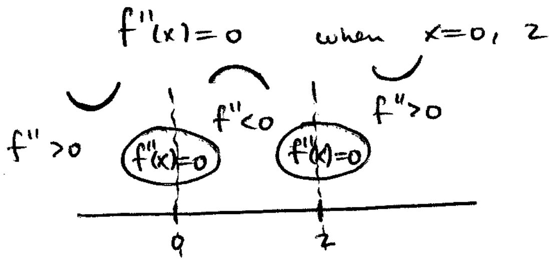


Thus, f is increasing on $(3, \infty)$ and is decreasing on $(-\infty, 3)$, with a horizontal tangent at $x=0$.

b. (*5 pts*) Determine the intervals on which f is concave upward or concave downward and find its inflection points.

(sol) $f''(x) = 12x^2 - 24x = 12x(x-2)$

$f''(x) = 0$ when $x = 0, 2$



Thus, f is concave upward on $(-\infty, 0)$ and $(2, \infty)$ and is concave downward on $(0, 2)$.

Inflection pts at $(0, 10)$ and $(2, -6)$

c. (*5 pts*) Find the absolute maximum and absolute minimum values of f on the interval $[-1, 4]$.

(sol)

X	-1	0	3	4
$f(x)$	15	10	-17	10
	Abs max		Abs. min	

The Absolute max. value is $f(-1) = 15$ and the absolute min. value is $f(3) = -17$.

5. (*15 points*) Find the limit:

a. (*5 pts*) $\displaystyle\lim_{x\to 0+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$

(sol)

$$\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0^+}\left(\frac{x-\sin x}{x\sin x}\right)\underset{\text{"}0/0\text{"}}{\overset{H}{=}}\lim_{x\to 0^+}\frac{1-\cos x}{\sin x+x\cos x}$$

$$\underset{\text{"}0/0\text{"}}{\overset{H}{=}}\lim_{x\to 0^+}\frac{\sin x}{2\cos x-x\sin x}=\frac{0}{2}=\boxed{0}$$

b. (*5 pts*) $\displaystyle\lim_{x\to 0+}(1+x)^{1/x}$

(sol) $y=(1+x)^{1/x}\ \Rightarrow\ \ln y=\frac{1}{x}\ln(1+x)$

$$\Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\frac{\ln(1+x)}{x}\overset{H}{=}\lim_{x\to 0^+}\frac{\frac{1}{1+x}}{1}=1$$

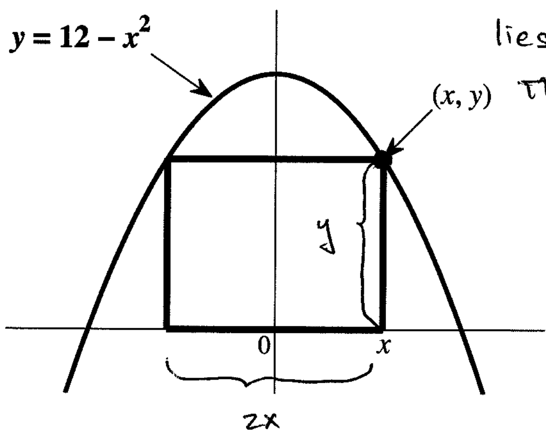
$$\therefore \lim_{x\to 0^+}(1+x)^{1/x}=\lim_{x\to 0^+}y=\lim_{x\to 0^+}e^{\ln y}=\boxed{e}$$

c. (*5 pts*) $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$

(sol)

$$\lim_{x\to\infty}\frac{e^x}{x^2}\underset{\text{"}\infty/\infty\text{"}}{\overset{H}{=}}\lim_{x\to\infty}\frac{e^x}{2x}\underset{\text{"}\infty/\infty\text{"}}{=}\lim_{x\to\infty}\frac{e^x}{2}=\boxed{\infty}$$

6. (*15 points*) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have and what are its dimensions?



(sol) Let (x, y) be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and y, so its area is

$A = 2xy$. : primary EQN

To express the primary EQN as a ftn of one variable, we use the fact that (x, y) lies on the parabola $y = 12 - x^2$.

Thus, $\boxed{A = 2x(12 - x^2)}$

The domain of this ftn is

$x \geq 0$ and $12 - x^2 \geq 0 \,(\Leftrightarrow -\sqrt{12} \leq x \leq \sqrt{12})$

, so $\boxed{0 \leq x \leq \sqrt{12}}$.

Its derivative is

$A'(x) = 24 - 6x^2 = 6(4 - x^2)$

and $A(x) = 0$ when $4 = x^2$, i.e., $x = 2$ $(\because x \geq 0)$

x	0	2	$\sqrt{12}$
$A(x)$	0	$4 \cdot (12 - 4) = 32$	0
		Abs. max.	

Thus, the area of largest inscribed rectangle is

$\boxed{A(2) = 32}$

and the dimensions are $\boxed{4 \text{ (width)} \times 8 \text{ (length)}}$.

7. (*15 points*) A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

(sol). Distance x and z are increasing, but distance y is decreasing; so $\frac{dy}{dt}$ is negative.

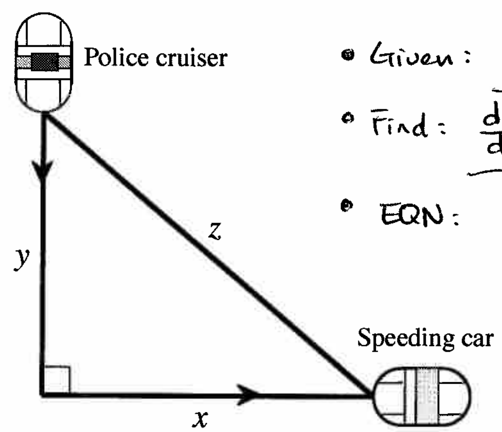


FIGURE 1. Let x be the distance of the speeding car from the intersection, let y be the distance of the police cruiser from the intersection, and let z be the distance between the car and the cruiser.

- Given: $\frac{dz}{dt} = 20\text{mph}$ when $\begin{cases} x = 0.8 \\ y = 0.6 \end{cases}$
- Find: $\frac{dx}{dt} = ?$ if $\frac{dy}{dt} = -60$ mph.
- EQN: $x^2 + y^2 = z^2$ (the Pythagorean TH.)

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$\Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{x}\left[z\frac{dz}{dt} - y\frac{dy}{dt}\right]$$

We now substitute the numerical values:

$$\begin{cases} x = 0.8 \\ y = 0.6 \\ z = \sqrt{x^2 + y^2} = \sqrt{(0.8)^2 + (0.6)^2} = 1 \\ \frac{dy}{dt} = -60 \\ \frac{dz}{dt} = 20 \end{cases}$$

Thus, $\frac{dx}{dt} = \frac{1}{0.8}\left[1 \cdot 20 - 0.6 \cdot (-60)\right]$

$$= \frac{1}{0.8}\left[20 + 36\right] = \frac{10}{\cancel{8}} \cdot \cancel{56}^{7} = 70.$$

At the moment in question, the car's speed is $\boxed{70\text{ mph}}$